

Power laws and stretched exponentials in a noisy finite-time-singularity model

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We discuss the influence of white noise on a generic dynamical finite-time-singularity model for a single degree of freedom. We find that the noise effectively resolves the finite-time-singularity and replaces it by a first-passage-time or absorbing state distribution with a peak at the singularity and a long time tail exhibiting power law or stretched exponential behavior. The study might be of relevance in the context of hydrodynamics on a nanometer scale, in material physics, and in biophysics.

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I. INTRODUCTION

There is a continuing current interest in the influence of noise on the behavior of nonlinear dynamical systems [1,2]. Here the issues, for example, associated with the interference of stochastic noise with the deterministic chaos of maps [3] or extended systems, as for example, the noise-driven Kuramoto-Shivashinski equation [4], are of fundamental interest.

In a particular class of systems the nonlinear character gives rise to finite-time-singularities, that is, solutions that cease to be valid beyond a particular finite-time span. One encounters finite-time singularities in stellar structure, turbulent flow, and bacterial growth [5–7] as well as in Euler flows and in free-surface flows [8–10]. Finite-time singularities are also encountered in modeling in econophysics, geophysics, and material physics [11–16].

In the context of hydrodynamical flow on a nanoscale [17], where microscopic degrees of freedom come into play, it is a relevant issue how noise influences the hydrodynamical behavior near a finite-time singularity. Leaving aside the issue of the detailed reduction of the hydrodynamical equations to a nanoscale and the influence of noise on this scale to further study, we assume in the present context that a single variable or “reaction coordinate” effectively captures the interplay between the singularity and the noise.

We thus propose to consider a simple generic model system with one degree of freedom governed by a nonlinear Langevin equation driven by Gaussian white noise,

$$\frac{dx}{dt} = -\frac{\lambda}{2|x|^{1+\mu}} + \eta, \quad \langle \eta \eta \rangle(t) = \Delta \delta(t). \quad (1.1)$$

The model is characterized by the coupling parameter λ , determining the amplitude of the singular term, the index $\mu \geq 0$, characterizing the nature of the singularity, and the

noise parameter Δ determining the strength of the noise correlations. Specifically, in the case of a thermal environment at temperature T the noise strength $\Delta \propto T$.

In the absence of noise this model exhibits a finite-time singularity at a time t_0 , where the variable x vanishes with a power law behavior determined by μ . When noise is added the finite-time-singularity event at t_0 becomes a statistical event and is conveniently characterized by a first-passage-time distribution $W(t)$ [18]. For zero noise we thus have $W(t) = \delta(t - t_0)$, restating the presence of the finite-time singularity. In the presence of noise $W(t)$ develops a peak about $t = t_0$, vanishes at short times, and acquires a long time tail.

The model in Eq. (1.1) has also been studied in the context of persistence distributions related to the nonequilibrium critical dynamics of the two-dimensional XY model [19] and in the context of non-Gaussian Markov processes [20]. Finally, regularized for small x , the model enters in connection with an analysis of long-range correlated stationary processes [21].

It follows from our analysis below that for $\mu = 0$, the logarithmic case, the distribution at long times is given by the power law behavior

$$W(t) \sim t^{-\alpha}, \quad \alpha = \frac{3}{2} + \frac{\lambda}{2\Delta}. \quad (1.2)$$

For vanishing nonlinearity, i.e., $\lambda = 0$, the finite-time singularity is absent and the Langevin equation (1.1) describes a simple random walk of the reaction coordinate, yielding the well-known exponent $\alpha = 3/2$ [18,22,23]. In the nonlinear case with a finite-time singularity the exponent attains a non-universal correction depending on the ratio of the nonlinear strength to the strength of the noise; for a thermal environment the correction is proportional to $1/T$.

In the generic case for $\mu > 0$ considered here we find that the falloff is slower and that the correction to the random walk result is given by a stretched exponential,

$$W(t) \sim t^{-3/2} \exp[A(t^{-\mu/(2+\mu)} - 1)], \quad (1.3)$$

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where $A \rightarrow \lambda/\Delta\mu$ for $\mu \rightarrow 0$; in the limit $\mu \rightarrow 0$ this expression reduces to expression (1.2). The above results for $\mu = 0$ and $\mu > 0$ have also been obtained in the context of the critical dynamics of the XY model [19].

The paper is organized in the following manner. In Sec. II we introduce the finite-time-singularity model in the noiseless case and discuss its properties. In Sec. III we consider the noisy case and discuss the ensuing Langevin equation and associated Fokker-Planck equation. We discuss the relationship to an absorbing state problem and introduce the first-passage-time or absorbing state distribution. In Sec. IV we review the weak noise WKB phase space approach to the Fokker-Planck equation, apply it to the finite-time-singularity problem, and discuss the associated dynamical phase space problem. In Sec. V we apply the WKB phase space approach and evaluate the weak noise absorbing state distribution at long times. We derive the random walk result in the linear case for $\lambda = 0$, the power law tail for $\mu = 0$, and the stretched exponential behavior for $\mu > 0$. In Sec. VI we derive an exact solution of the Fokker-Planck equation in the case $\mu = 0$ in terms of a Bessel function and present an expression for the absorbing state distribution. In Sec. VII we present a summary and a conclusion. Details of the phase space method is discussed in Appendix A; aspects of the exact solution in Appendix B.

II. FINITE-TIME-SINGULARITY MODEL

Let us first consider the noiseless case for $\Delta = 0$. It is instructive to express the equation of motion (1.1) in the form

$$\frac{dx}{dt} = -\frac{1}{2} \frac{dF}{dx}, \quad (2.1)$$

where the potential or free energy has the form,

$$F(x) = \lambda \ln|x| \quad \text{for } \mu = 0, \quad (2.2)$$

$$F(x) = -\frac{\lambda}{\mu} |x|^{-\mu} \quad \text{for } \mu > 0. \quad (2.3)$$

The free energy has a logarithmic sink for $\mu = 0$ and a power law sink for general $\mu > 0$. In both cases F drives x to the absorbing state $x = 0$. Solving Eq. (2.1) in the logarithmic case we obtain for positive x the solution

$$x = \sqrt{\lambda} \sqrt{t_0 - t}. \quad (2.4)$$

This solution displays a finite-time singularity at $t_0 = x_0^2/\lambda$, where x_0 is the initial value at time $t = 0$, with x approaching the absorbing state with exponent $1/2$. In other words, the attraction to the sink in the free energy occurs in a finite time span; for times beyond t_0 Eq. (2.1) does not possess a real solution. This is the way we define a finite-time singularity in the present context. For general $\mu > 0$ we obtain the generalization of the solution (2.4),

$$x = [\lambda(2 + \mu)/2]^{1/(2 + \mu)} [t_0 - t]^{1/(2 + \mu)}, \quad (2.5)$$

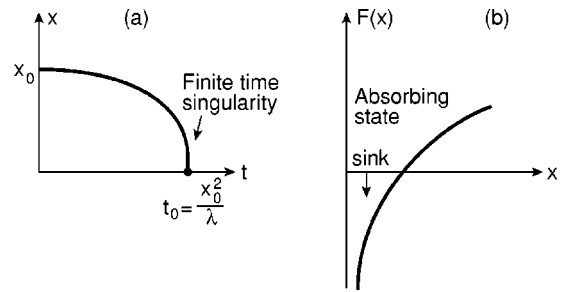


FIG. 1. In (a) we show the time evolution of the single degree of freedom x . x reaches the absorbing state $x=0$ at a finite time t_0 . In (b) we depict the free energy $F(x)$ driving the equation. The absorbing state $x=0$ corresponds to the sink in $F(x)$.

which approach the absorbing state with exponent $1/(2 + \mu)$ at time $t_0 = 2x_0^{2+\mu}/\lambda(2 + \mu)$. In Fig. 1 we have for $\mu = 0$ depicted the solution x and the free energy $F(x)$ driving x to zero.¹

III. LANGEVIN AND FOKKER-PLANCK EQUATIONS

In the presence of noise the finite-time-singularity problem is characterized by the Langevin equation (1.1). Here the noise drives the variable x into a fluctuating state in competition with the free energy that tends to drive x to the absorbing state $x = 0$. In the case $\mu \geq 0$ treated here with an absorbing state the free energy has a sink and there is no stationary distribution; the probability leaks out at $x = 0$.²

From another point of view, introducing the variable $y = x^{2+\mu}$ the Langevin equation (1.1) takes the form

$$\frac{1}{2 + \mu} \frac{dy}{dt} = -\frac{\lambda}{2} + y^{1/(2 + \mu)} \eta. \quad (3.1)$$

In the noiseless case the variable y decreases linearly to zero at time $t = t_0$. In the presence of noise y fluctuates. We note, however, that the noise is manifestly quenched at $y = 0$, yielding the absorbing state. Absorbing state models of the type in Eq. (3.1) for extended systems have been studied extensively in the context of directed percolation, catalysis, and Reggeon field theory [24–27].

In order to analyze the stochastic aspects of finite-time singularities in the presence of noise we need the time-dependent probability distribution $P(x, t)$ and the derived first-passage-time or absorbing state probability distribution $W(t)$. The distribution $P(x, t)$ is defined according to [23,28]

¹For $\mu < 0$ the free energy vanishes at $x = 0$. For $\mu < -2$ x possesses a run-away solution at a finite time. In the present context we confine our discussion to the case $\mu > 0$, where the free energy has a sink and x approaches an absorbing state.

²In the case $\mu = -2$ the free energy, $F = (\lambda/2)x^2$, has the form of a harmonic potential and the resulting Langevin equation, $dx/dt = -(\lambda/2)x + \eta$, governs the stochastic dynamics of an overdamped noise-driven harmonic oscillator. The stationary distribution is given by the Boltzmann factor $P_0 \propto \exp(-F/T)$, where the effective temperature according to the fluctuation-dissipation theorem is $T = \Delta$. For $-2 < \mu < 0$ the same result holds with the free energy given by Eq. (2.3).

$$P(y,t) = \langle \delta(y-x(t)) \rangle, \quad (3.2)$$

where x is a stochastic solution of Eq. (1.1) and $\langle \dots \rangle$ indicates an average over the noise η driving x . In the absence of noise $P(y,t) = \delta(y-x(t))$, where x is the deterministic solution of Eq. (2.1) given by Eqs. (2.4) and (2.5) and depicted in Fig. 1. At time $t=0$ the variable x evolves from the initial condition x_0 , implying the boundary condition

$$P(x,0) = \delta(x-x_0). \quad (3.3)$$

At short times x is close to x_0 and the singular term is not yet operational. In this regime we then obtain ordinary random walk with the Gaussian distribution

$$P(x,t) = (2\pi\Delta t)^{-1/2} \exp\left[-\frac{(x-x_0)^2}{2\Delta t}\right], \quad (3.4)$$

approaching Eq. (3.3) for $t \rightarrow 0$. At longer times the barrier $\lambda/2x^{1+\mu}$ comes into play preventing x from crossing the absorbing state $x=0$. This is, however, a random event that can occur at an arbitrary time instant, i.e., the finite-time singularity taking place at t_0 in the deterministic case is effectively resolved in the noisy case. For not too large noise strength the distribution is peaked about the noiseless solution and vanishes for $x \rightarrow 0$, corresponding to the absorbing state, implying the boundary condition

$$P(0,t) = 0. \quad (3.5)$$

In order to model a possible experimental situation the first-passage-time or here absorbing state distribution $W(t)$ is of more direct interest [28,29]. First-passage properties in fact underlie a large class of stochastic processes such as diffusion limited growth, neuron dynamics, self-organized criticality, and stochastic resonance [18].

Since $P(0,t) = 0$ for all t due to the absorbing state, the probability that x is not reaching $x=0$ in time t is thus given by $\int_0^\infty P(x,t) dx$, implying that the probability $-dW$ that x does reach $x=0$ in time t is $-dW = -\int_0^\infty dx dt (dP/dt)$. Consequently, the absorbing state distribution $W(t)$ is determined by the expression [23]

$$W(t) = - \int_0^\infty \frac{\partial P(x,t)}{\partial t} dx. \quad (3.6)$$

In the absence of noise $P(x,t) = \delta(x-x(t))$ and it follows from Eq. (3.6) that $W(t) = \delta(t-t_0)$, in accordance with the finite-time singularity at $t=t_0$. For weak noise we anticipate that $W(t)$ will peak about t_0 with vanishing tails for small t and large t . In Fig. 2 we have depicted a particular realization of x in the noisy case, the distribution $P(x,t)$ in a plot versus x and t , and the absorbing state distribution $W(t)$.

In the case of Gaussian white noise the distribution $P(x,t)$ satisfies the Fokker-Planck equation [28,29]

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x} \left[-\frac{dF}{dx} P + \Delta \frac{\partial P}{\partial x} \right], \quad (3.7)$$

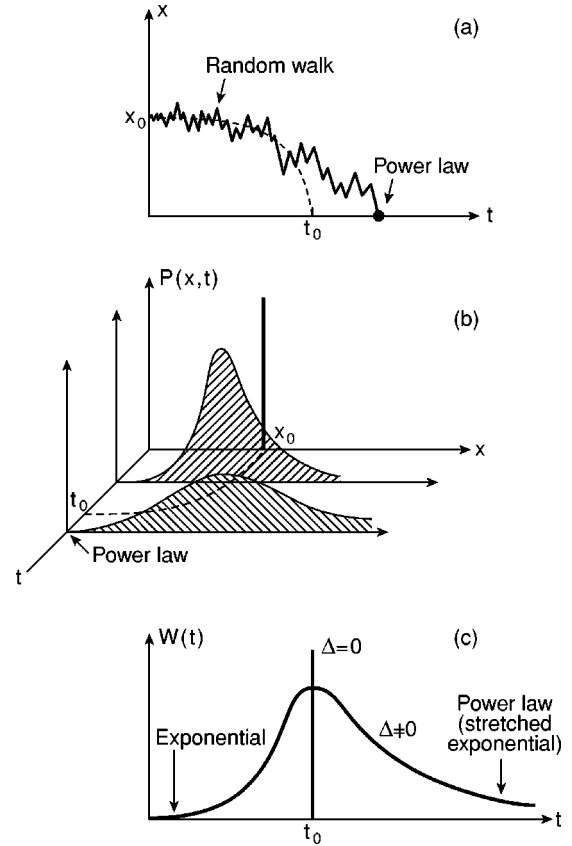


FIG. 2. In (a) we show a particular realization of x . At early times near x_0 we have random walk behavior. At longer times x is attracted to the absorbing state at $x=0$. In (b) we depict the distribution $P(x,t)$ in the logarithmic case for $\mu=0$. For $t=0$ $P(x,0) = \delta(x-x_0)$, i.e., the initial condition. For larger t the distribution is broadened about the noiseless trajectory. $P(x,t)$ exhibits a power law behavior for large t near the absorbing state. In (c) we show the absorbing state distribution $W(t)$. For small t the distribution is exponentially small; for large t it displays a power law behavior for $\mu=0$ and a stretched exponential behavior in the generic case for $\mu>0$.

in the present case subject to the boundary conditions $P(x,0) = \delta(x-x_0)$ and $P(0,t) = 0$. The absorbing state distribution $W(t)$ then follows from Eq. (3.6). We note that the Fokker-Planck equation has the form of a conservation law $\partial P/\partial t + \partial J/\partial x = 0$, defining the probability current $J = (1/2)(dF/dx)P - (1/2)\Delta \partial P/\partial x$. Inserting Eq. (3.7) in the expression (3.6) for the distribution $W(t)$ and using that $J \rightarrow 0$ for $x \rightarrow \infty$ we obtain another expression for $W(t)$:

$$W(t) = \frac{1}{2} \left[\Delta \frac{\partial P}{\partial x} - P \frac{dF}{dx} \right]_{x=0}. \quad (3.8)$$

The absorbing state distribution is thus equal to the probability current at the absorbing state.

IV. WKB PHASE SPACE APPROACH

From a structural point of view the Fokker-Planck equation (3.7) has the form of an imaginary-time Schrödinger

equation $\Delta \partial P / \partial t = HP$, driven by the Hamiltonian or Liouillian H . The noise strength Δ then plays the role of an effective Planck constant with P corresponding to the wave function. A method utilizing a nonperturbative WKB phase space approach to a generic Fokker-Planck equation for extended system was derived in the context of the Kardar-Parizi-Zhang equation describing interface growth [30–32]. In the case of a single degree of freedom this method amounts to the eikonal approximation [2,23,29], see also [33,34]. For systems with many degrees of freedom the method has for example been expounded in Ref. [35], based on the functional formulation of the Langevin equation [36,37]. In the present formulation [30–32] the emphasis is on the canonical phase space analysis and the use of dynamical system theory [38,39], for more details we refer to Appendix A.

The weak noise WKB approximation corresponds to the ansatz $P \propto \exp[-S/\Delta]$. The weight function or action S then to leading asymptotic order in Δ satisfies a Hamilton-Jacobi equation $\partial S / \partial t + H = 0$ which in turn implies a *principle of least action* and Hamiltonian equations of motion [40,41]. In the present context the Hamiltonian has the form

$$H = \frac{1}{2}p^2 - \frac{1}{2}p \frac{dF}{dx}, \quad (4.1)$$

yielding the equations of motion

$$\frac{dx}{dt} = -\frac{1}{2} \frac{dF}{dx} + p, \quad (4.2)$$

$$\frac{dp}{dt} = \frac{1}{2}p \frac{d^2F}{dx^2}, \quad (4.3)$$

replacing the Langevin equation (1.1). The noise η is then represented by the momentum $p = \partial S / \partial x$ conjugate to x . The Eqs. (4.2) and (4.3) determine orbits in a canonical phase space spanned by x and p . Since the system is conserved the orbits lie on the constant energy manifold(s) given by $E = H$. The free energy F is given by Eqs. (2.2) and (2.3) and the action associated with an orbit from x_0 to x in time t has the form

$$S(x_0 \rightarrow x, t) = \int_0^t dt \left[p \frac{dx}{dt} - H \right]. \quad (4.4)$$

According to the ansatz the probability distribution is then given by

$$P(x, t) \propto \exp \left[-\frac{S(x_0 \rightarrow x, t)}{\Delta} \right]. \quad (4.5)$$

The zero-energy manifold $E=0$ plays an important role in determining the long time distributions. Inserting $dF/dx = \lambda/x^{1+\mu}$ in Eq. (4.1) the zero-energy manifold has a submanifold structure given by $p=0$ and $p = \lambda/x^{1+\mu}$. According to Eq. (4.2) the $p=0$ submanifold corresponds to the noiseless deterministic motion given by Eq. (2.1). In Fig. 3 we have depicted the canonical phase space spanned by x

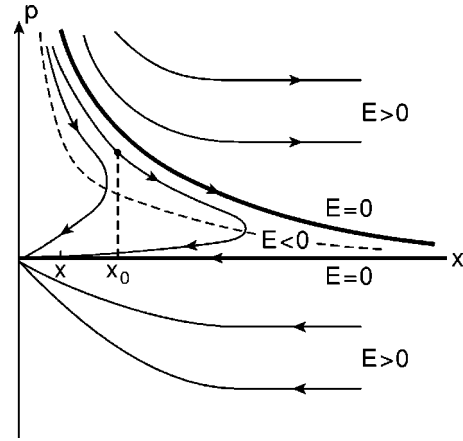


FIG. 3. We show the topology of phase space. The bold lines indicate the zero-energy submanifolds. We show representative orbits for $E > 0$ and $E < 0$. The dashed line indicates a nullcline ($dx/dt=0$) to the saddle point $(x, p) = (\infty, 0)$. At long times the orbit from x_0 to x migrates to the zero-energy submanifolds.

and p . The heavy lines represent the zero-energy submanifolds $p=0$ and $p = \lambda/x^{1+\mu}$. For $E > 0$ the energy surfaces are equidistant for large x approaching a constant p value; for small x the manifold $p \sim \lambda/x^{1+\mu}$ for $p > 0$ and $p \sim -4Ex^{1+\mu}/\lambda$ for $p < 0$. For $E < 0$ the energy surfaces are confined between the zero-energy submanifolds; the manifolds approach $(x, p) = (0, 0)$ according to $p \sim 4|E|x^{1+\mu}/\lambda$ and for large p as $p \sim \lambda/x^{1+\mu}$. For $E \rightarrow -\infty$ the orbits approach the positive p half-axis. The arrows indicate the direction of motion on the manifolds. The dashed line indicates a nullcline ($dx/dt=0$) passing through the hyperbolic fixed point $(x, p) = (\infty, 0)$. In the long time limit the orbit from x_0 to x converges towards the zero-energy submanifolds.

V. RANDOM WALK AND LONG TIME TAILS

The weak noise phase space approach reviewed above affords a simple derivation of the asymptotic long time behavior of the distributions for the finite-time-singularity problem. In order to derive the transition probability $P(x, t)$ according to Eq. (4.5) we simply have to identify the relevant orbit in phase space from x_0 to x which at long times passes close to the zero-energy manifolds.

A. The random walk case

It is instructive first to consider the case $\lambda=0$. Here the finite-time singularity at $x=0$ is absent, there is no absorbing state and the Langevin equation (1.1) takes the form $dx/dt = \eta(t)$, describing random walk on the whole axis. The Hamiltonian is given by $H = (1/2)p^2$ and the equations of motion (4.2) and (4.3) have the form $dx/dt = p$, $dp/dt = 0$ with solutions $p = p_0$ and $x = x_0 + p_0 t$. Inserting dx/dt and H in Eq. (4.4) for the action we obtain $S = (1/2) \int_0^t dt p^2 = (1/2) p_0^2 t$ and finally using Eq. (4.5) the Gaussian distribution (3.4) for random walk.

We note that in order to obtain the correct limit of the finite-time-singularity problem we must incorporate the ab-

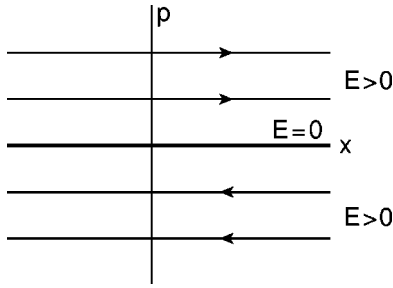


FIG. 4. We depict the phase space topology in the random walk case for $\lambda=0$. The energy manifolds are equidistant corresponding to constant values of p . The bold line for $p=0$ is the zero-energy manifold.

absorbing state condition at $x=0$. This is achieved by using the method of mirrors [23] and considering for $P(x,t)$ the linear combination,

$$P(x,t) = (2\pi\Delta t)^{-1/2} \left(\exp\left[-\frac{(x-x_0)^2}{2\Delta t}\right] - \exp\left[-\frac{(x+x_0)^2}{2\Delta t}\right] \right), \quad (5.1)$$

in the half space $x \geq 0$. This distribution is a solution of the Fokker-Planck equation and vanishes for $x=0$. For small x it behaves linearly with x ,

$$P(x,t) = (2/\pi)^{1/2} x (\Delta t)^{-3/2} \exp(-x_0^2/2\Delta t). \quad (5.2)$$

Using Eq. (3.8) we readily obtain the well-known random walk result

$$W(t) = (2/\pi)^{1/2} x_0 (\Delta t)^{-3/2} \exp(-x_0^2/2\Delta t). \quad (5.3)$$

For small t the distribution vanishes exponentially. It displays a maximum at $t_0 = x_0^2/3\Delta$ and falls off algebraically as $t^{-\alpha}$ for large t with scaling exponent $\alpha=3/2$. This behavior is in accordance with the general discussion in Sec. III and is graphically depicted in Fig. 2. The phase space topology in the random walk case is shown in Fig. 4.

B. The absorbing state case

For zero noise and for $\lambda=0$ we have from Eq. (1.1) $x = x_0$ at all times, whereas the solution in the case of a finite-time singularity is attracted to the absorbing state at time t_0 . In the presence of noise the attraction gives rise to a change of the form of the absorbing state distribution $W(t)$ from a δ function peak to a broadened peak. For large λ the distribution shows a maximum about t_0 ; for intermediate values of λ the maximum is between t_0 and the random walk value $x_0^2/3\Delta$. Because of the attraction to the absorbing state we also obtain a faster long time fall-off and thus a positive correction to the random walk exponent $\alpha=3/2$, depending on the strength λ .

The plot in Fig. 3 permits a simple qualitative discussion of the finite-time-singularity phase space phenomenology. Firstly, a short time orbit from x_0 to x corresponds to large

value of $|p|$. In this region and for not too small x the phase space topology in Fig. 3 is similar to the random walk case depicted in Fig. 4 and we infer unbiased random walk behavior, yielding the expressions (5.1) and (5.3). Second, searching for longer time orbits from x_0 to x we must choose smaller p and we move into the region of phase space for negative energy where the finite-time singularity dominates the topology, as shown in Fig. 3. In the limit of long times the orbit approaches asymptotically the zero-energy submanifolds.

1. The logarithmic case $\mu=0$

In the logarithmic case $\mu=0$ the zero-energy condition using Eqs. (4.1) and (2.2) yields the relationship $p=\lambda/x$, corresponding to the hyperbolic manifold; note that the $p=0$ manifold corresponds to deterministic motion and yields $S=0$. Setting $p=\lambda/x$ and $H=0$ in the action in Eq. (4.4) we then have $S=\int^t dt p dx/dt = \int^x dx \lambda/x = \lambda \ln x$. Moreover, for $p=\lambda/x$ the equation of motion (4.2) reduces to $dx/dt = \lambda/2x$ with the growing solution $x^2 = \lambda t$. In the long time limit where the orbit is close to the zero-energy manifold we thus obtain $S \sim (\lambda/2) \ln t$, yielding according to Eq. (4.5) the power law distribution $P(x,t) \propto t^{-(\lambda/2\Delta)}$. Owing to the absorbing state the distribution $P(x,t)$ must vanish for $x \rightarrow 0$. As discussed in Appendix A this limit is reproduced by pushing the WKB approximation in Sec. IV to next order in Δ , yielding the correction $-\Delta \ln x$ to S . Finally, we obtain in the weak noise long time limit the distribution

$$P(x,t) \propto x t^{-(\lambda/2\Delta)}. \quad (5.4)$$

Moreover, applying Eq. (3.8) we deduce the weak noise long time absorbing state distribution

$$W(t) \propto t^{-(\lambda/2\Delta)}, \quad (5.5)$$

with scaling exponent $\alpha = \lambda/2\Delta$. The expressions (5.4) and (5.5) show that the finite-time singularity or, equivalently, absorbing state attracts the random walker and increase the falloff exponent α . We note that the WKB approximation to leading order in Δ fails to retrieve the random walk exponent 3/2 in the limit $\lambda \rightarrow 0$ and the maximum of $W(t)$ about t_0 .

2. The generic case $\mu > 0$

In the generic case $\mu > 0$ the zero-energy manifold implies the constraint $p = \lambda/x^{1+\mu}$ and we obtain similar to the logarithmic case above the action $S = \int^t dt p dx/dt = \int^x dx \lambda/x^{1+\mu} = -(\lambda/\mu)x^{-\mu} + \text{const}$ and the equation of motion, $dx/dt = \lambda/2x^{1+\mu}$, on the zero-energy manifold with solution $x^{\mu+2} = (1+\mu/2)\lambda t$. In the long time limit we thus find the action $S = -(\lambda/\mu)[(1+\mu/2)\lambda t]^{-\mu/(2+\mu)} + \lambda/\mu$. The second-order correction to S , evaluated in Appendix A, is given by $-\Delta(1+\mu)\ln x$ and we obtain the weak noise long time distribution

$$P(x,t) \propto x^{1+\mu} \exp\left(\frac{\lambda}{\Delta\mu} \left[\left\{ \left(1 + \frac{\mu}{2}\right) \lambda t \right\}^{-\mu/(2+\mu)} - 1 \right]\right), \quad (5.6)$$

and absorbing state distribution

$$W(t) \sim \exp\left(\frac{\lambda}{\Delta\mu} \left[\left\{ \left(1 + \frac{\mu}{2}\right)\lambda t \right\}^{-[\mu/(2+\mu)]} - 1 \right]\right). \quad (5.7)$$

The expressions (5.6) and (5.7) show that in the case of a generic finite-time singularity characterized by the index μ the power law behavior of $P(x,t)$ and $W(t)$ is altered to a stretched exponential behavior depending on μ . In the limit $\mu \rightarrow 0$ we obtain the power law behavior. We note again that the WKB approximation is unable to produce the random walk prefactor $t^{-3/2}$ for $\lambda=0$ and the peak of $W(t)$ about t_0 .

VI. SOLUTION OF THE FOKKER-PLANCK EQUATION

In this section we return to the Fokker-Planck equation (3.7) and present exact expressions for the transition probability $P(x,t)$ and the absorbing state distribution $W(t)$ in the logarithmic case $\mu=0$. We have summarized key points in the derivation here and defer details to Appendix B.

A. Quantum particle in a repulsive potential

In the logarithmic case the Fokker-Planck equation assumes the form

$$\frac{\partial P}{\partial t} = \frac{\Delta}{2} \frac{\partial^2 P}{\partial x^2} + \frac{\lambda}{2x} \frac{\partial P}{\partial x} - \frac{\lambda}{2x^2} P. \quad (6.1)$$

Removing the first-order term by means of the gauge transformation $\exp(h) = x^{-\lambda/2\Delta}$ we have

$$-\Delta \frac{\partial}{\partial t} [\exp(-h)P] = H[\exp(-h)P], \quad (6.2)$$

where the Hamiltonian H is given by

$$H = -\frac{1}{2}\Delta^2 \frac{\partial^2}{\partial x^2} + \frac{\lambda^2}{8} \left[1 + \frac{2\Delta}{\lambda} \right] \frac{1}{x^2}. \quad (6.3)$$

This Hamiltonian describes the motion of a unit mass quantum particle in one dimension subject to a centrifugal barrier of strength $(\lambda^2/8)(1 + 2\Delta/\lambda)$. For $\lambda=0$ the barrier is absent and the particle can move over the whole axis; this case corresponds to ordinary random walk [23]. For $\lambda \neq 0$ the particle cannot cross the barrier and is confined to either half-space; this corresponds to the case of a finite-time singularity subject to noise and an absorbing state at $x=0$.

The Fokker-Planck equation (6.2) has the form of an imaginary time Schrödinger equation with Planck constant Δ for the wave function $\exp(-h)P$ and is readily analyzed in terms of Bessel functions [42]. Incorporating the initial condition $P(x,0) = \delta(x-x_0)$ by defining $P(x,t) \rightarrow P(x,t)\theta(t)$ we obtain the inhomogeneous differential equation

$$\frac{\partial P(x,t)}{\partial t} = \delta(x-x_0)\delta(t) - \frac{\Delta}{2} e^h H e^{-h} P(x,t) \quad (6.4)$$

for the determination of $P(x,t)$. On the positive x and k axis the wave functions $\psi_k(x) = (kx)^{1/2} J_{1/2+\lambda/2\Delta}(kx)$ form, ac-

cording to the Fourier-Bessel transform [43], an orthonormal and complete set satisfying the eigenvalue equation $H\psi_k(x) = (\Delta^2/2)k^2\psi_k(x)$. Expanding the right hand side of Eq. (6.4) on the set ψ_k and using a well-known identity for Bessel functions [44,45] we obtain for the probability distribution $P(x,t)$ the following closed expression:

$$P(x,t) = \frac{x^{(\lambda/2\Delta)+(1/2)} e^{-(x^2+x_0^2)/2\Delta t}}{x_0^{(\lambda/2\Delta)-(1/2)} \Delta t} I_{1/2+(\lambda/2\Delta)}\left(\frac{xx_0}{\Delta t}\right). \quad (6.5)$$

Here I_ν is the Bessel function of imaginary argument, $I_\nu(z) = (-i)^\nu J_\nu(iz)$ [43].

By means of Eq. (3.8) we moreover deduce the absorbing state distribution

$$W(t) = \frac{2x_0^{1+\lambda/\Delta}}{\Gamma(1/2+\lambda/2\Delta)} e^{-x_0^2/2\Delta t} (2\Delta t)^{-(3/2)-(\lambda/2\Delta)}. \quad (6.6)$$

Similar expressions have also been derived in the context of the XY model [19].

B. The distribution $P(x,t)$

The expression (6.5) provides the complete solution of the finite-time-singularity problem for $\mu=0$. The expression is discussed in more detail in Appendix B. For $t=0$ we have $P(x,0) = \delta(x-x_0)$ in accordance with the initial condition (3.3). For small t we obtain the random walk result $P(x,t) = \exp[-(x-x_0)^2/2\Delta t]/(2\pi\Delta t)^{1/2}$ in accordance with Eq. (3.4). For $\lambda=0$ we have $P(x,t) = \{\exp[-(x-x_0)^2/2\Delta t] + x_0 \rightarrow -x_0\}/(2\pi\Delta t)^{1/2}$ in agreement with Eq. (5.1) for random walk with an absorbing wall at $x=0$.

For long times and x close to the absorbing state $x=0$ we obtain the asymptotic form

$$P(x,t) \propto \frac{2xx_0^{1+\lambda/\Delta} e^{-(x^2+x_0^2)/2\Delta t}}{\Gamma\left(\frac{3}{2} + \frac{\lambda}{2\Delta}\right)} (2\Delta t)^{-(3/2)-(\lambda/2\Delta)}. \quad (6.7)$$

For small x the distribution vanishes linearly due to the absorbing state. For large t the distribution exhibits a power law behavior with scaling exponent $\alpha = 3/2 + \lambda/2\Delta$. In the weak noise limit the distribution is peaked about the noiseless solution (2.4). For $\lambda=0$ we obtain the random walk result in Eq. (5.2) and we note that the finite-time singularity or absorbing state lead to an increase of the scaling exponent and thus a faster falloff in time. In the weak noise limit the scaling exponent approaches $\lambda/2\Delta$ in agreement with the WKB analysis in Sec. IV. In Fig. 2 we have depicted the distribution $P(x,t)$.

C. The distribution $W(t)$

The absorbing state distribution $W(t)$ in Eq. (6.6) vanishes exponentially for small t . For large t the distribution shows a power law behavior with exponent $\alpha = 3/2 + \lambda/2\Delta$.

In the deterministic limit $\Delta=0$ we have $W(t)=\delta(t-t_0)$, where $t_0=x_0^2/\lambda$. For small Δ the exponent approaches $\lambda/2\Delta$ in accordance with Eq. (5.5). The distribution has a maximum at

$$t_{\max}=\frac{x_0^2}{3\Delta+\lambda}. \quad (6.8)$$

For $\Delta=0$ we have $t_{\max}\rightarrow t_0$ and for $\lambda=0$ the random walk result $t_{\max}=x_0/3\Delta$. For large coupling strength $t_{\max}\rightarrow 0$. Expanding $W(t)$ about t_{\max} we obtain the Gaussian distribution

$$W(t)\propto e^{-(t-t_{\max})^2/\sigma^2}, \quad (6.9)$$

characterized by the mean square width

$$\sigma^2=\frac{4\Delta x_0^4}{(3\Delta+\lambda)^3}. \quad (6.10)$$

Since $W(t)$ falls off as a power of t only a finite number of moments $\langle t^n \rangle = \int t^n W(t) dt$ exists. For $(2n-1)\Delta < \lambda$ we have

$$\langle t^n \rangle = \prod_{p=1}^n \left(\frac{x_0^2}{\lambda - (2p-1)\Delta} \right). \quad (6.11)$$

The distribution $W(t)$ is shown in Fig. 2.

VII. SUMMARY AND CONCLUSION

In this paper we have addressed the problem of the influence of white Gaussian noise of strength Δ on a generic finite-time singularity of strength λ , characterized by the exponent μ . We have for simplicity considered only a single degree of freedom. We have found that in the case of a logarithmic sink in the free energy driving the variable, corresponding to a square root singularity, the first-passage-time or absorbing state distribution $W(t)$ displays a peak about the finite-time singularity and a long time power law tail $\propto t^{-\alpha}$, characterized by the scaling exponent $\alpha=3/2+\lambda/2\Delta$. The exponent is nonuniversal and depends on the ratio between the singularity strength λ and the noise strength Δ . In the case where the noise originates from a thermal environment at temperature T we have $\Delta \propto T$ and the scaling exponent depends on the temperature, $\alpha=3/2+\text{const}/T$.

In the generic case of a finite-time singularity characterized by the exponent $\mu > 0$ the weak noise WKB approach shows that the power law tail for $\mu=0$ is changed to a stretched exponential with a slower falloff.

To the extent that the character of a finite-time singularity in the vicinity of threshold can be modeled with a single degree of freedom the present study should hold as regard the influence of noise on the time distribution. We note, in particular, that in the case of a thermal environment at temperature T the change of the scaling exponent becomes large in the limit of low temperatures as the distribution narrows around the noiseless threshold time.

The present study also suggests generalizations to the case of damping and to the case of several coupled variable subject to a finite-time singularity.

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APPENDIX A: THE PHASE SPACE METHOD

The weak noise WKB approximation applied to the Fokker-Planck equation is well documented [1,23,29,34]. Here we review this method with emphasis on a canonical phase space approach which we have found useful in discussing the pattern formation and scaling in the noisy Burgers equation [30,32,46–48]. We also note that the approach follows from a saddle point approximation to the functional Martin-Siggia-Rose approach to nonlinear Langevin equations [36,37,49,50].

1. To leading order Δ

Taking as our starting point a generic Langevin equation for one degree of freedom x driven by Gaussian white noise,

$$\frac{dx}{dt} = -\frac{1}{2}G(x) + \eta(t), \quad \langle \eta\eta \rangle(t) = \delta(t), \quad (A1)$$

the associated Fokker-Planck equation for the distribution $P(x,t)$ takes the form

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x} \left[GP + \Delta \frac{\partial P}{\partial x} \right]. \quad (A2)$$

Applying like in quantum mechanics [42] the WKB approximation,

$$P(x,t) \propto \exp \left[-\frac{S(x,t)}{\Delta} \right], \quad (A3)$$

and expanding the action S in powers of the noise strength Δ , $S=S_0+\Delta S_1$, S_0 satisfies the Hamilton-Jacobi equation [40]

$$\frac{\partial S_0}{\partial t} + H(p,x) = 0, \quad p = \frac{\partial S_0}{\partial x}, \quad (A4)$$

implying a *principle of least action* [40]. The action S_0 thus has the form

$$S_0(x,t) = \int dt \left(p \frac{dx}{dt} - H \right) \quad (A5)$$

with Hamiltonian given by

$$H = \frac{1}{2}p(p - G), \quad (\text{A6})$$

implying the Hamilton equations of motion

$$\frac{dx}{dt} = -\frac{1}{2}G + p, \quad (\text{A7})$$

$$\frac{dp}{dt} = \frac{1}{2}p \frac{dG}{dx}. \quad (\text{A8})$$

The deterministic coupled Eqs. (A7) and (A8) replace the stochastic Langevin equation (1.1) for weak noise. The noise η is replaced by the canonically conjugate momentum p , which by means of Eqs. (A3) and (A4) is given in terms of the distribution P

$$p = -\Delta \partial \ln P / \partial x. \quad (\text{A9})$$

The equations of motion define orbits in a (x, p) phase space lying on the constant energy surfaces $E = H$ and the general discussion of the original stochastic problem is replaced by an analysis of the phase space topology. The prescription for deriving the distribution to leading order in Δ thus amounts to (i) solve the equations of motion (A7) and (A8) for an orbit from an initial value x_0 to a final value x reached in the time span t , p being a slaved variable, (ii) evaluate the action S associated with an orbit according to Eq. (A5), and, finally, (iii) derive the transition probability from x_0 to x in time t using the ansatz (A3). The zero-energy manifold here plays an important role in determining the long time distributions. In the limit $t \rightarrow \infty$ a given finite-time orbit from x_0 to x thus converges to the zero-energy manifold.

In the case of an overdamped harmonic oscillator described by the Langevin equation $dx/dt = -\omega x + \eta$ the phase space analysis was carried out in Ref. [30]. In the case of random walk given by $dx/dt = \eta$ the analysis is performed in Sec. V A.

Finite-time-singularity case

In the generic finite-time-singularity case, $G = \lambda/x^{1+\mu}$, and we have the Hamiltonian $H = (1/2)p(p - \lambda/x^{1+\mu})$, yielding the equations of motion $dx/dt = -\lambda/2x^{1+\mu} + p$ and $dp/dt = -\lambda(1+\mu)p/2x^{2+\mu}$. These equations are, however, not particularly tractable and we therefore only consider the logarithmic case for $\mu = 0$. Here $G = \lambda/x$ and we obtain the Hamiltonian

$$H = \frac{1}{2}p \left(p - \frac{\lambda}{x} \right), \quad (\text{A10})$$

and the equations of motion

$$\frac{dx}{dt} = -\frac{\lambda}{2x} + p, \quad (\text{A11})$$

$$\frac{dp}{dt} = -\frac{\lambda}{2} \frac{p}{x^2}. \quad (\text{A12})$$

These equations have a hyperbolic fixed point at $(x, p) = (\infty, 0)$. The nullcline $dx/dt = 0$ to the saddle point is given by $p = \lambda/2x$, indicated in Fig. 3 by the dashed line. The zero-energy manifolds are given by $p = 0$ and $p = \lambda/x$. The conserved energy $H = E$ provides the first constant of integration. Solving for x we have $x = \lambda/(p^2 - 2E)$. For $E > 0$ $p \rightarrow \pm(2E)^{1/2}$ for $x \rightarrow \infty$ as indicated in Fig. 3, yielding in that limit the random walk phase space topology depicted in Fig. 4. For $E < 0$ $x = \lambda/(p^2 + 2|E|)$ exhibiting a maximum at the nullcline in a plot of x versus p as shown for two representative orbits in Fig. 3. Using energy conservation to solve the equation of motion for p and subsequently for x we obtain the solutions $x^2 = (t + t_1)[\lambda + 2E(t + t_1)]$ and $p^2 = 2E + \lambda/(t + t_1)$, where t_1 is the second constant of integration. A specific orbit from x_0 to x_1 in time t thus determines the constants E and t_1 ; the momentum p becomes a slaved variable, and the action evaluated along the orbit yields the distribution.

Considering as final value the absorbing state $x_1 = 0$, the long time orbits lie in the negative energy region and eliminating E we obtain the solution, $0 < t' < t$

$$x^2 = (x_0^2(1 - t'/t) + \lambda t')(1 - t'/t). \quad (\text{A13})$$

The energy is given by $|E| = (\lambda t - x_0^2)/t^2$ and we note that the energy approaches zero in the long time limit, i.e., the orbit from x_0 to $x = 0$ migrates to the zero-energy manifold. Finally, for the action associated with the orbit we obtain

$$S_0 = \frac{1}{2} \left[\lambda \ln \frac{\lambda t}{x_0^2} - 1 \right], \quad (\text{A14})$$

yielding the long time distribution

$$P(x_0 \rightarrow 0, t) \propto t^{-\lambda/2\Delta}, \quad (\text{A15})$$

in accordance with the expression (5.4).

Alternatively, eliminating the momentum p the equations of motion (A11) and (A12) reduce to a second-order equation for x , $d^2x/dt^2 = -dV/dx$, describing the motion of a particle of unit mass in the attractive potential $V(x) = -(1/8)\lambda^2/x^2$. It then follows by simple quadrature that all direct orbits to the absorbing state $x = 0$ take a finite time, whereas the traversal time of negative-energy orbits with a turning point diverges in the limit $|E| \rightarrow 0$; this is in accordance with the phase space behavior shown in Fig 3.

2. Next leading order in Δ

The next leading order in Δ is obtained from S_1 which by insertion satisfies the equation of motion,

$$-\frac{\partial S_1}{\partial t} = \left(\frac{\partial S_0}{\partial x} - \frac{G}{2} \right) \frac{\partial S_1}{\partial x} + \frac{1}{2} \frac{d}{dx} \left(G - \frac{\partial S_0}{\partial x} \right), \quad (\text{A16})$$

where $\partial S_0 / \partial x$ is obtained from the first-order solution.

a. Random walk case

From the random walk case discussed in Sec. V A we have $S_0 = (x - x_0)^2/2t$. Consequently, Eq. (A16) takes the form

$$-\frac{\partial S_1}{\partial t} = \left(\frac{x - x_0}{t} \right) \frac{\partial S_1}{\partial x} - \frac{1}{2}t, \quad (\text{A17})$$

with a particular time-dependent solution

$$S_1 = \frac{1}{2} \ln|t|, \quad (\text{A18})$$

yielding $S = (x - x_0)^2/t + (\Delta/2) \ln|t|$ and the Gaussian distribution

$$P(x, t) \propto |t|^{-1/2} \exp\left[-\frac{(x - x_0)^2}{\Delta t} \right]. \quad (\text{A19})$$

As in the quantum case [42] the next leading correction yields the normalization factor $|t|^{-1/2}$.

b. Finite-time-singularity case

In the finite-time-singularity case $G = \lambda/x^{1+\mu}$ and from above $\partial S_0/\partial x = 0$. We then obtain inserting in Eq. (A16)

$$-\frac{\partial S_1}{\partial t} = -\frac{\lambda}{2x^{1+\mu}} \frac{\partial S_1}{\partial x} - \frac{1+\mu}{2} \frac{\lambda}{x^{2+\mu}}, \quad (\text{A20})$$

with a particular space-dependent solution

$$S_1 = -(1+\mu) \ln x, \quad (\text{A21})$$

giving rise to the factor $x^{1+\mu}$ in Eq. (5.6).

APPENDIX B: THE FOKKER PLANCK EQUATION

Here we discuss the Fokker-Planck equation (6.1) in the logarithmic case $\mu = 0$ in more detail. Applying the gauge transformation, $\exp(h) = x^{-\lambda/2\Delta}$, and incorporating the boundary condition (3.3), $P(x, 0) = \delta(x - x_0)$, we obtain the inhomogeneous differential equation (6.4), with Hamiltonian H given by Eq. (6.3) corresponding to the motion of a quantum particle subject to a centrifugal barrier $\propto 1/x^2$.

1. Exact solution

The right hand side of Eq. (6.1) has the same form as the standard Bessel equation [44,45]. Noting also the analogy to the quantum case of particle motion in spherical coordinates [42] it follows that

$$\psi_k(x) = (kx)^{1/2} Z_\nu(kx), \quad (\text{B1})$$

where $Z_\nu(kx)$ is a solution of the Bessel equation, satisfies the eigenvalue equation

$$H\psi_k(x) = k^2 \psi_k(x) \quad (\text{B2})$$

for $\nu = \pm(1/2 + \lambda/2\Delta)$. The Bessel function of the first kind $J_\nu(kx)$ satisfies the absorbing state boundary condition J_ν

$\rightarrow 0$ for $x \rightarrow 0$ and the completeness and orthogonality of $\psi_k(x)$ follow from the Fourier-Bessel integral representation [44]

$$f(r) = \int_0^\infty dk \int_0^\infty dr' k J_\nu(kr) J_\nu(kr') f(r'), \quad (\text{B3})$$

valid for $\nu > 1/2$. We proceed by Fourier transforming (6.4),

$$P(x, t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} p_\omega(x), \quad (\text{B4})$$

and subsequently expanding $p_\omega(x) x^{\lambda/2\Delta}$ and $\delta(x - x_0)$ on the eigenfunctions $\psi_k(x)$,

$$p_\omega(x) x^{\lambda/2\Delta} = \int_0^\infty dk \psi_k(x) p_{\omega k}, \quad (\text{B5})$$

$$\delta(x - x_0) = \int_0^\infty dk \psi_k(x) \psi_k(x_0), \quad (\text{B6})$$

yielding the expansion coefficients

$$p_{\omega k} = \frac{x_0^{\lambda/2\Delta} \psi_k(x_0)}{-i\omega + \Delta k^2/2}. \quad (\text{B7})$$

Finally, integrating over ω in the lower half-plane and picking up contributions from the branch cut we obtain

$$P(x, t) = \int_0^\infty dk k e^{-\Delta k^2 t/2} (xx_0)^{1/2} (x/x_0)^{\lambda/2\Delta} \times J_{1/2+\lambda/2\Delta}(kx) J_{1/2+\lambda/2\Delta}(kx_0). \quad (\text{B8})$$

This integral can be reduced further using the identity [45]

$$\int_0^\infty dx x e^{-\rho^2 x^2} J_\nu(\alpha x) J_\nu(\beta x) = \frac{1}{2\rho^2} e^{(\alpha^2 + \beta^2)/4\rho^2} I_\nu(\alpha\beta/2\rho^2), \quad (\text{B9})$$

yielding Eq. (6.5) in Sec. VI.

2. Random walk, short time, and long time limits

In the random walk case for $\lambda = 0$ we obtain using $I_{1/2} = 2(1/2\pi x)^{1/2} \sinh x$, the expression (5.1). In the short time limit $t \ll xx_0/\Delta$, using $I_\nu(x) \propto (1/2\pi x)^{1/2} \exp(x)$ for $x \rightarrow \infty$ [44] we obtain Eq. (3.4) and for $t = 0$ the boundary condition (3.3).

In the long time limit $t \gg xx_0/\Delta$ using $I_\nu(x) \propto (x/2)^\nu/\Gamma(\nu+1)$ for $x \rightarrow 0$ [44] we obtain Eq. (6.7) Using Eq. (3.8) and $\Gamma(z+1) = z\Gamma(z)$ we finally obtain the absorbing state distribution (6.6).

The moments of $W(t)$ are easily worked out. Using $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ [44,45] we have

$$\langle t^n \rangle = \int t^n W(t) dt = \left(\frac{x_0^2}{2\Delta} \right)^n \frac{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2\Delta} - n\right)}{\Gamma\left(\frac{1}{2} + \frac{\lambda}{2\Delta}\right)}, \quad (\text{B10})$$

or further reduced for $(2n-1)\Delta < \lambda$ the expression (6.11).

3. Weak noise limit

In the limit $\Delta \rightarrow 0$ the distribution $P(x,t)$ is centered about the noiseless solution (2.4). In terms of the exact solution (6.5) this is a singular limit since both order and argument in $I_\nu(x)$ diverge. Using the spectral representation [44,45]

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi \cosh(x \cos \theta) \sin^{2\nu} \theta d\theta, \quad (\text{B11})$$

introducing the variable u according to

$$\sinh u = \frac{\lambda t}{2xx_0}, \quad (\text{B12})$$

defining

$$f_\pm(\theta) = \ln \sin \theta + \frac{1}{2} \left(1 \pm \frac{\cos \theta}{\sinh u} \right), \quad (\text{B13})$$

and using

$$\Gamma\left(1 + \frac{\lambda}{2\Delta}\right)\Gamma\left(\frac{1}{2}\right) \approx \pi\sqrt{2}e^{-\lambda/2\Delta} \left(\frac{\lambda}{2\Delta}\right)^{1/2 + \lambda/2\Delta}, \quad (\text{B14})$$

for small Δ we obtain by insertion in Eq. (6.5)

$$P(x,t) \approx \frac{1}{4\pi\sqrt{2}} \frac{1}{x_0} \frac{\lambda}{\Delta} \left(\frac{x_0^2}{\lambda t}\right)^{(1/2) + (\lambda/2\Delta)} e^{-(x^2 + x_0^2)/2\Delta t} \times \int_0^\pi d\theta \frac{\sin \theta}{\sinh u} [e^{(\lambda/\Delta)f_+(\theta)} + e^{(\lambda/\Delta)f_-(\theta)}]. \quad (\text{B15})$$

The expression (B15) for $P(x,t)$ is directly amenable to an asymptotic analysis for $\Delta \rightarrow 0$ by means of Laplace's method [51]. For small Δ the main contributions to the integral originate from the maxima of $f_+(\theta)$ and $f_-(\theta)$. The two maxima in the interval $0 < \theta < \pi$ are given by $\cos \theta_\pm = \pm \exp(-u)$, yielding $f'_\pm(\theta_\pm) = -\coth u$ and $f_\pm(\theta_\pm) = (1/2)[\ln(1 - e^{-2u}) + \coth u]$. Performing the Gaussian integrals about the maxima we thus obtain the asymptotic result valid for small Δ and fixed u , i.e., fixed x/t .

$$P(x,t) \approx \left(\frac{\lambda}{2\pi\Delta}\right)^{1/2} \frac{1}{x_0} \left(\frac{x_0^2}{\lambda t}\right)^{(1/2) + (\lambda/2\Delta)} \times \frac{(1 - e^{-2u})^{(1/2) + (\lambda/2\Delta)}}{(\sinh 2u)^{1/2}} \exp\left(-\frac{F(x,t)}{2\Delta t}\right), \quad (\text{B16})$$

where $F(x,t) = x^2 + x_0^2 - \lambda t \coth u$ or by insertion

$$F(x,t) = x^2 + x_0^2 - [(2xx_0)^2 + (\lambda t)^2]^{1/2}. \quad (\text{B17})$$

In the short time limit $\lambda t \ll 2xx_0$ u is small and $\sinh u \sim u$, i.e., $u \sim \lambda t/2xx_0$ and we obtain $P(x,t) \sim (1/2\pi\Delta t)^{1/2} \exp[-(x-x_0)^2/2\Delta t]$, yielding $\delta(x-x_0)$ for $t = 0$. The weak coupling limit $\lambda \rightarrow 0$ for fixed x and t is also consistent; we obtain $P(x,t) \sim (1/2\pi\Delta t)^{1/2} \exp[-(x-x_0)^2/2\Delta t]$. For weak noise the peak of the distribution is determined by the condition $F=0$, yielding $x = (x_0^2 - \lambda t)^{1/2}$ and the peak thus follows the noiseless finite-time-singularity solution (2.4).

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